Math 245C Lecture 11 Notes

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1 Properties of Convolution and Young's Inequality

1.1 Properties of convolution

Proposition 1.1. *The convolution satisfies the following properties:*

1. $f * g = g * f$
2. $(f * g) * h = f * (g * h)$
3. *If* $z \in \mathbb{R}^n$, $\tau_x(f * g) = (\tau_z f) * g = f * (\tau_z g)$.
4. *If* $A = \{x + y : x \in \operatorname{supp}(f), y \in \operatorname{supp}(g)\}$, *then* $\operatorname{supp}(f * g) \subseteq \overline{A}$.

Proof. Let $f, g : \mathbb{R}^n \to \mathbb{R}$.

1. We have already proved this.
2. Let $x \in \mathbb{R}^n$. We have

$$\begin{aligned}(f * g) * h(x) &= \int_{\mathbb{R}^n} (f * g)(x - y)h(y)\, dy \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x - y - z)g(z)h(y)\, dy\, dz\end{aligned}$$

But

$$f * (g * h)(x) = \int_{\mathbb{R}}^n f(x - u)g * h(u)\, du = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x - u)g(u - v)h(v).$$

So set $u = y$ and $u - v = z$. Then

$$x - y - z = x - v - (u - v) = x - u,$$

so the two expressions are equal.

3. We have

$$\begin{aligned}\tau_z(f*g)(z) &= f*g(x-z)\\ &= \int_{\mathbb{R}^n} f(x-z-y)g9y)\,dy\\ &= \int_{\mathbb{R}^n} (\tau_z f)(x-y)g(y)\,dy\\ &= (\tau_z f)*g(x).\end{aligned}$$

Since $f*g = g*f$, we conclude that

$$\tau_z(f*g) = \tau_z(g*f) = (\tau_z g)*f = f*(\tau_z g).$$

4. Assume $x \notin A$. Observe that

$$f(x-y)g(y) = \begin{cases} 0 & y \notin \operatorname{supp}(g) \\ 0 & y \notin \operatorname{supp}(g). \end{cases}$$

Hence, $f*g(x) = 0$. Then $A^c \subseteq \{f*g = 0\}$, so $\{f*g \neq 0\} \subseteq A$, which makes $\overline{\{f*g \neq 0\}} \subseteq \overline{A}$. □

1.2 Young's inequality

Our goal is that if $1 \leq p, q, r < \infty$ and $r^{-1} + 1 = p^{-1} + q^{-1}$, then

$$\|f*g\|_r \leq \|f\|_p \|g\|_q.$$

It is important to note here that this bound is independent of the dimension.

Theorem 1.1 (Young's inequality)**.** *Let $1 \leq q \leq \infty$, let $f \in L^1$, and let $g \in L^q$. For a.e. $x \in \mathbb{R}^n$, $f*g(x)$ exists, and*

$$\|f*g\|_q \leq \|f\|_1 \|g\|_q.$$

Proof. Assume $q < \infty$.

$$\begin{aligned}\|f*g\|_q &= \left(\int_{\mathbb{R}^n} |f*g(x)|^q\,dx \right)^{1/q}\\ &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(y)g(x-y)\,dy \right|^q dx \right)^{1/q}\end{aligned}$$

Use Minkowski's inequality.

$$\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(y)g(x-y)|^q\,dx \right)^{1/q} dy$$

$$= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} |g(x-y)|^q \, dx \right)^{1/q} dy$$

Set $z = x - y$.

$$= \int_{\mathbb{R}^n} |f(y)| \|g\|_q \, dy = \|f\|_1 \|g\|_q.$$

If $q = \infty$, the proof is simpler. □

Definition 1.1. $C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : \{|f| \geq \varepsilon\}$ is compact $\forall \varepsilon > 0\}$ is the set of functions that **vanish at** ∞.

Remark 1.1. As a subspace of L^∞, $\overline{C_c(\mathbb{R}^n)} = C_0(\mathbb{R}^n)$.

Theorem 1.2. *Let $1 \leq p, q, \leq \infty$ be conjugate exponents. Let $f \in L^p$ and $g \in L^q$. Then*

1. *$f * g(x)$ exists for each $x \in \mathbb{R}^n$, and*

$$|f * g| \leq \|f\|_p \|g\|_q.$$

2. *$f * g$ is uniformly continuous.*

3. *If $1 < p < \infty$, then $f * g \in C_0(\mathbb{R}^n)$.*

Proof. For $p \neq \infty$, by Hölder's inequality,

$$|f * g(x)| = \left| \int_{\mathbb{R}^n} f(x-y) g(y) \, dy \right| \leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p \, dy \right)^{1/p} \|g\|_q = \|f\|_p \|g\|_q.$$

If $p = \infty$, the proof is easier.

To prove the second statement, it suffices to show that $\lim_{y \to 0} \|\tau_y (f * g) - f * g\|_u = 0$. □

We will finish the proof next time.